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Comonotone Polynomial Approximation

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Let P_n be the set of all algebraic polynomials of degree n or less. For $f \in C[a, b]$, the degree of approximation to f by polynomials in P_n is $E_n(f) = \inf\{\| f - p \|: p \in P_n\}$, where the norm is the uniform norm. Jackson's theorem [1] states that there exists $C > 0$ such that $E_n(f) \leqslant C\omega(f; 1/n)$, where $\omega(f; \delta)$ is the modulus of continuity of f.

f is said to be piecewise monotone if it has only a finite number of local maxima and minima in $[a, b]$. The local maxima and minima in (a, b) are called the peaks of f. Wolibner [5] has shown that for any $\epsilon > 0$ there exists a polynomial, p, such that $\| f - p \| < \epsilon$ and p is comonotone with f; i.e., p increases and decreases simultaneously with f. Let $E_n^*(f) = \inf\{\| f - p \|: p \in P_n, p$ comonotone with $f\}$. Clearly $E_n^*(f) \geqslant E_n(f)$. We seek an upper bound on $E_n^*(f)$. For f monotone, Lorentz and Zeller [3] have shown that there exists $C_1 > 0$ such that $E_n^*(f) \leqslant C_1\omega(f; 1/n)$. Newman, Passow, and Raymon [4] have obtained results of a modified nature. They have shown that there exists $p \in P_n$ satisfying $\| f - p \| < C_2\omega(f; 1/n)$, C_2 an absolute constant, such that f and p are comonotone except in certain neighborhoods (whose diameters tend to zero with n) of the peaks. In this note we obtain a comonotone approximation on the entire interval $[a, b]$, but at a sacrifice in the accuracy of approximation.

LEMMA 1. *Let $f \in C^{(j+1)}[a, b]$ and suppose that $f(u) = 0$, $u \in (a, b)$. Let $g(x) = f[x, u]$, the divided difference of f, where we define $f[u, u] = f'(u)$. Then $g \in C^j[a, b]$ and $\| g^{(j)} \| \leqslant (j + 1)^{-1} \| f^{(j+1)} \|$.*

Proof. $g(x) = f[x, u] = \int_0^1 f'((x-u)t+u)\,dt$, [2, p. 250]. Thus,

$$g'(x) = \int_0^1 tf''((x-u)t+u)\,dt,$$
$$\vdots$$
$$g^{(j)}(x) = \int_0^1 t^j f^{(j+1)}((x-u)t+u)\,dt.$$

Therefore $g \in C^j[a, b]$ and $\| g^{(j)} \| \leqslant \| f^{(j+1)} \| \int_0^1 t^j\,dt = (j+1)^{-1} \| f^{(j+1)} \|$.

LEMMA 2. *Let f be a piecewise monotone function, with peaks at $x_1, x_2, \ldots, x_k$, and suppose that $f \in C^{(j+k+1)}[a, b]$. Let*

$$g(x) = \sum_{i=1}^{k} \left[\prod_{\substack{l=1\\ l\neq i}}^{k} (x_i - x_l) \right]^{-1} f'[x, x_i], \qquad \text{where} \quad f'[x_i, x_i] = f''(x_i).$$

Then

(a) $g \in C^{(j+k-1)}[a, b]$;

(b) $\| g^{(j+k-1)} \| \leqslant \dfrac{\| f^{(j+k+1)} \|}{(j+k)} \displaystyle\sum_{i=1}^{k} \left[\prod_{\substack{l=1\\ l\neq i}}^{k} | x_i - x_l | \right]^{-1}$;

(c) $\| g^{(j)} \| \leqslant \dfrac{j!}{(j+k)!} \| f^{(j+k+1)} \|$.

Proof. Since $f' \in C^{(j+k)}[a, b]$, by Lemma 1, $f'[x, x_i] \in C^{(j+k-1)}[a, b]$, so that $g \in C^{(j+k-1)}[a, b]$, proving (a).

Now

$$\begin{aligned} g(x) &= \sum_{i=1}^{k} \left[\prod_{\substack{l=1\\ l\neq i}}^{k} (x_i - x_l) \right]^{-1} f'[x, x_i] \\ &= \sum_{i=1}^{k} \left[\prod_{\substack{l=1\\ l\neq i}}^{k} (x_i - x_l) \right]^{-1} \int_0^1 f''((x - x_i)t + x_i)\,dt. \end{aligned}$$

Therefore,

$$g^{(j+k-1)}(x) = \sum_{i=1}^{k} \left[\prod_{\substack{l=1\\ l\neq i}}^{k} (x_i - x_l) \right]^{-1} \int_0^1 t^{(j+k-1)} f^{(j+k+1)}((x - x_i)t + x_i)\,dt,$$

and (b) follows from this.

To prove (c), let $g_1(x) = f'[x, x_1]$ and $g_{i+1}(x) = g_i[x, x_{i+1}]$, $i = 1, 2,..., k-1$. Then $g_{i+1}(x) = f'[x, x_1, ..., x_{i+1}]$, $i = 1, 2,..., k-1$, [2, p. 248]. Hence,

$$g_k(x) = f'[x, x_1, ..., x_k] = \sum_{i=1}^{k} \left[\prod_{\substack{l=1 \\ l\neq i}}^{k} (x_i - x_l)\right]^{-1} f'[x, x_i] \tag{1}$$

the last expression being equal to $g(x)$ [2, p. 255]. By Lemma 1, $g_1 \in C^{(j+k-1)}[a, b]$ and $\| g_1^{(j+k-1)} \| \leqslant (j+k)^{-1} \| f^{(j+k+1)} \|$, $g_2 \in C^{(j+k-2)}[a, b]$ and $\| g_2^{(j+k-2)} \| \leqslant (j+k-1)^{-1} \| g_1^{(j+k-1)} \|,..., g_k \in C^j[a, b]$ and $\| g_k^{(j)} \| \leqslant (j+1)^{-1} \| g_{k-1}^{(j+1)} \|$.

Thus

$$\| g^{(j)} \| = \| g_k^{(j)} \| \leqslant \left[\prod_{l=1}^{k} (j+l)\right]^{-1} \| f^{(j+k+1)} \| = \frac{j!}{(j+k)!} \| f^{(j+k+1)} \|,$$

and the proof of the lemma is complete.

THEOREM 1. *Let f be a piecewise monotone function with peaks at $x_1, x_2, ..., x_k$, and suppose that $f \in C^{(j+k+1)}[a, b]$. Then there exists d_j such that, for $n > 2(k+j)$,*

$$E_n^*(f) \leqslant \frac{d_j(b-a)^{k+1} \| f^{(j+k+1)} \|}{n^j}.$$

Proof. Define g as in Lemma 2, and note from (1) that

$$g(x) = \frac{f'(x)}{\prod_{i=1}^{k} (x - x_i)},$$

since $f'(x_i) = 0$ for $i = 1, 2,..., k$. Thus g maintains a constant sign on $[a, b]$, which, we may assume, is nonnegative. Therefore, there exists $q \in P_{n-k-1}$ such that $q(x) \geqslant 0$ on $[a, b]$ and $\| g - q \| \leqslant 2E_{n-k-1}(g)$. Hence,

$$\left| \frac{f'(x)}{\prod_{i=1}^{k} (x - x_i)} - q(x) \right| \leqslant 2E_{n-k-1}(g),$$

so that

$$\left| f'(x) - q(x) \prod_{i=1}^{k} (x - x_i) \right| \leqslant 2(b-a)^k E_{n-k-1}(g).$$

Thus

$$\left| f(x) - f(a) - \int_a^x q(t) \prod_{i=1}^{k} (t - x_i)\, dt \right| \leqslant 2(b-a)^{k+1} E_{n-k-1}(g).$$

If we let

$$p(x) = f(a) + \int_a^x q(t) \prod_{i=1}^{k} (t - x_i)\, dt,$$

then $p \in P_n$, p is comonotone with f, and $\| f - p \| \leqslant 2(b - a)^{k+1} E_{n-k-1}(g)$. Since $g \in C^j[a, b]$ and $\| g^{(j)} \| \leqslant \| f^{(j+k+1)} \|$, there exists a_j such that

$$E_{n-k-1}(g) \leqslant \frac{a_j \| f^{(j+k+1)} \|}{(n-k)(n-k-1) \cdots (n-k-j+1)} \quad \text{for} \quad n > (k+j), \quad [1],$$

$$\leqslant \frac{2a_j \| f^{(j+k+1)} \|}{n^j} \quad \text{for} \quad n > 2(k+j).$$

Thus

$$\| f - p \| \leqslant \frac{4a_j(b-a)^{k+1} \| f^{(j+k+1)} \|}{n^j} = \frac{d_j(b-a)^{k+1} \| f^{(j+k+1)} \|}{n^j} \quad \text{for} \quad n > 2(k+j).$$

THEOREM 2. *Let f satisfy the hypotheses of Theorem 1. Then there exists $r_{j,k}$ such that, for $n > 4(k + j + 2)$,*

$$E_n^*(f) \leqslant \frac{(b-a)^{k+1} r_{j,k} \| f^{(j+k+1)} \|}{n^{j+k-1}},$$

where $r_{j,k}$ depends on $x_1, x_2, \ldots, x_k$ and j.

The proof of Theorem 2 is identical to that of Theorem 1, but makes use of parts (a) and (b) of Lemma 2 in the same way that Theorem 1 uses part (c) of that lemma.

Notice that the order of comonotone approximation in Theorem 2 is smaller than that in Theorem 1. On the other hand, the constant $r_{j,k}$ in Theorem 2 depends upon the location of the peaks of f, while the constant d_j in Theorem 1 is independent of f, n, and k.

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